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Slide polynomials

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Abstract. The fundamental slide basis of polynomials was recently introduced by the authors. We survey positivity properties of this basis, and applications to the important Schubert and key bases of polynomials.

Résumé. La base fondamentale des polynômes a été récemment introduite par les auteurs. Nous étudions les propriétés de positivité de cette base, et les applications à les bases des polynômes importants de Schubert et des clés.

Keywords: fundamental slide polynomials, Schubert polynomials, key polynomials, fundamental quasisymmetric functions, quasi-Schur functions, Kohnert tableaux

1 Introduction

Symmetric and quasisymmetric polynomials have many bases with interesting combinatorial properties and relationships to one another, and many beautiful and powerful tools and models have been developed to explain their combinatorics. In contrast, for the full polynomial ring, the combinatorial theory is much sparser and less developed. We aim to lift bases and models from symmetric and quasisymmetric polynomials to the full polynomial ring, as a means of better understanding important bases of polynomials such as the Schubert and key polynomials.

$$\begin{tikzcd} \mathfrak{S}_w \arrow[r] & \kappa_a \arrow[r] & \mathfrak{Q}_a \arrow[r] & \mathfrak{F}_a \\ & s_\lambda \arrow[ul] \arrow[u] \arrow[r] & QS_\alpha \arrow[u] \arrow[r] & F_\alpha \arrow[u] \end{tikzcd}$$

Figure 1: A right arrow from f to g indicates that f expands positively into the basis $\{g\}$, and an up arrow from f to g indicates that f is contained in the basis $\{g\}$.

The fundamental slide polynomials $\{\mathfrak{F}_a\}$, introduced by the authors in [2], are a basis of polynomials that lifts Gessel's fundamental basis of quasisymmetric polynomials

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[8]. Fundamental slide polynomials exhibit several desirable positivity properties: both Schubert and key polynomials expand positively in this basis, and we give positive combinatorial formulas for these expansions. Moreover, the fundamental slide basis has positive structure constants: we give Littlewood-Richardson rules for these numbers, and also for the fundamental slide expansion of products of Schubert polynomials. For key polynomials, we introduce the model of Kohnert tableaux [1] to define the new quasi-key basis $\{\mathfrak{Q}_a\}$ of polynomials, which lifts the quasi-Schur basis [9] of quasisymmetric polynomials, and to give positive combinatorial formulas for the expansion of key polynomials into quasi-key polynomials, and quasi-key polynomials into fundamental slide polynomials. Figure 1 shows how our new bases fit into the existing picture.

2 Slide polynomials and Schubert polynomials

We first define the new fundamental slide basis of polynomials [2]. For a a weak composition of length n, let x^a denote the monomial $x_1^{a_1} \cdots x_n^{a_n}$, and let $\mathrm{flat}(a)$ be the (strong) composition obtained by removing all 0 terms. Given weak compositions a, b of length n, we say that b *dominates* a, denoted by $b \geqslant a$, if $b_1 + \cdots + b_i \geqslant a_1 + \cdots + a_i$ for all $i = 1, \ldots, n$. Note that this extends the usual dominance order on partitions. A composition β *refines* a composition α if α can be obtained by summing consecutive entries of β, for example, $(2,1,2)$ refines $(2,3)$ but $(1,2,2)$ does not.

Definition 2.1 ([2]). For a weak composition a of length n, define the *fundamental slide polynomial* $\mathfrak{F}_a = \mathfrak{F}_a(x_1, \ldots, x_n)$ by

$$\mathfrak{F}_a = \sum_{\substack{b \geqslant a \\ \mathrm{flat}(b) \text{ refines } \mathrm{flat}(a)}} x^b, \tag{2.1}$$

For example, we have

$$\begin{aligned} \mathfrak{F}_{(0,2,0,3)} \quad = \quad & x^{0203} + x^{0230} + x^{2003} + x^{2030} + x^{2300} + x^{0212} + x^{1112} + x^{2012} + x^{2102} \\ & + x^{2111} + x^{2120} + x^{0221} + x^{2021} + x^{1121} + x^{2201} + x^{2210} + x^{1103} + x^{1130}. \end{aligned} \tag{2.2}$$

Notice that $\mathfrak{F}_{(0,2,0,3)}$ is not quasisymmetric in (x_1, x_2, x_3, x_4): it uses the monomial $x_2^2 x_4^3$ but does not use $x_3^2 x_4^3$. However, $\{\mathfrak{F}_a\}$ contains Gessel's fundamental basis $\{F_\alpha\}$ of quasisymmetric functions [8]: if the nonzero entries of a occur in an interval with last nonzero entry in position k, $\mathfrak{F}_a$ is equal to $F_\alpha(x_1, \ldots, x_k)$, where α is the nonzero entries of a read from left to right ([2, Lemma 3.8]).

The Schubert polynomials $\mathfrak{S}_w$, indexed by permutations, were introduced by Lascoux and Schützenberger [11]. These polynomials form a $\mathbb{Z}$-basis of polynomials and represent Schubert basis classes in the cohomology of the flag manifold. For more on Schubert polynomials, see [13]. We give the combinatorial definition due to [4, 3, 7].

A *(reduced) pipe dream* is a tiling of the first quadrant of $\mathbb{Z} \times \mathbb{Z}$ with *elbows* and finitely many *crosses* $+$ such that no two lines, or *pipes*, cross more than once. The *shape* of a pipe dream is the permutation obtained by following the pipes from the y-axis to the x-axis. Let $\mathrm{PD}(w)$ denote the set of pipe dreams of shape w. For example, the pipe dreams of shape $135264 \in \mathcal{S}_6$ are given Figure 2.

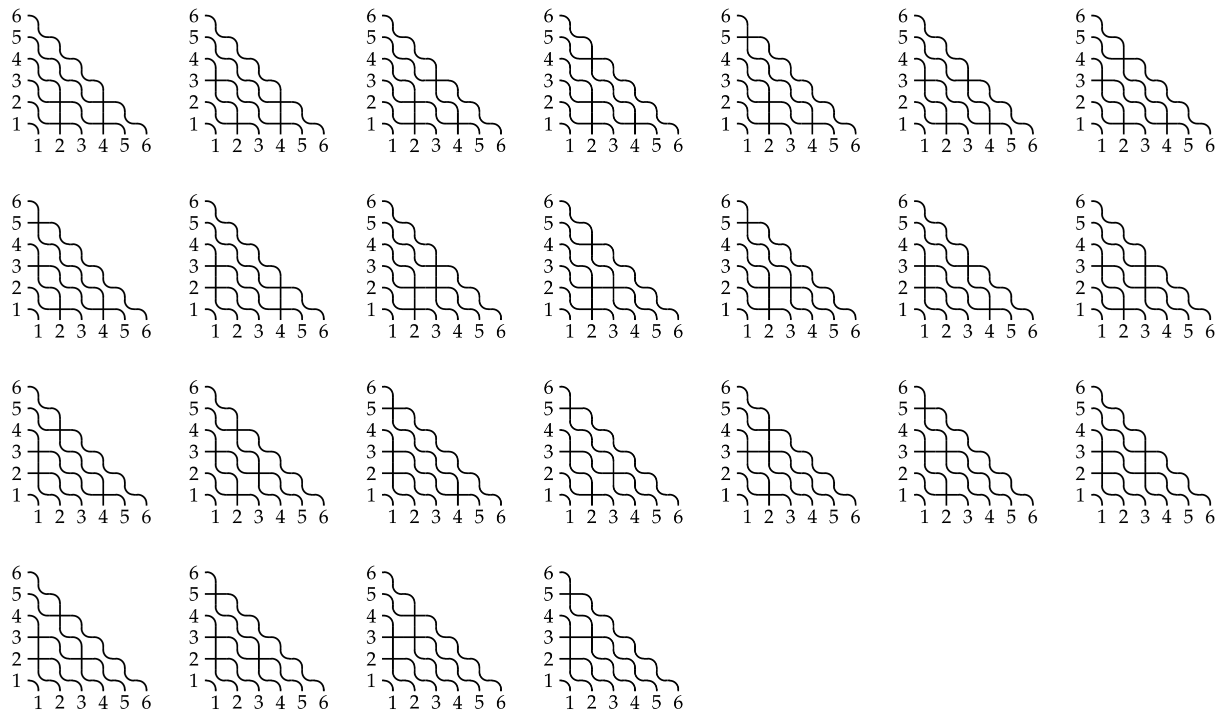

Figure 2: The 25 elements of PD(135264).

To each pipe dream P we associate the weak composition $\mathrm{wt}(P)$, where $\mathrm{wt}(P)_i$ is the number of crosses in the ith row of P.

Definition 2.2 ([3])**.** For w a permutation with no descents at or beyond n, the *Schubert polynomial* $\mathfrak{S}_w = \mathfrak{S}_w(x_1, \ldots, x_{n-1})$ is given by

$$\mathfrak{S}_w = \sum_{P \in \mathrm{PD}(w)} x^{\mathrm{wt}(P)}. \tag{2.3}$$

For example, from Figure 2 we can compute

$$\begin{aligned} \mathfrak{S}_{135264} = \; & x^{22000} + 2x^{21100} + x^{21010} + x^{21001} + x^{20200} + x^{20110} + x^{20101} \\ & + 2x^{12100} + x^{12010} + x^{12001} + 2x^{11200} + 2x^{11110} + 2x^{11101} + x^{10210} \\ & + x^{10201} + x^{02200} + x^{02110} + x^{02101} + x^{01210} + x^{01201}. \end{aligned} \tag{2.4}$$

Definition 2.3. ([2]) A pipe dream is *quasi-Yamanouchi* if, for every i, the westernmost $+$ in row i is in the first column or lies weakly west of some $+$ in row $i+1$. Let $\mathrm{QPD}(w)$ denote the set of quasi-Yamanouchi pipe dreams of shape w.

For example, of the pipe dreams for 135264 shown in Figure 2, five are quasi-Yamanouchi. These are shown in Figure 3.

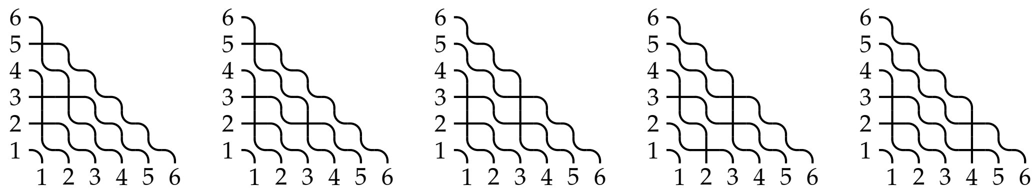

Figure 3: The five quasi-Yamanouchi pipe dreams for $w = 135264$.

Quasi-Yamanouchi pipe dreams index the fundamental slide expansion of Schubert polynomials.

Theorem 2.4. *([2]) For w any permutation, we have*

$$\mathfrak{S}_w = \sum_{P \in \mathrm{QPD}(w)} \mathfrak{F}_{\mathrm{wt}(P)}. \tag{2.5}$$

For example, the fundamental slide expansion of $\mathfrak{S}_{135264}$ has only five terms, corresponding to the five quasi-Yamanouchi pipe dreams of Figure 3:

$$\mathfrak{S}_{135264} = \mathfrak{F}_{(0,1,2,0,1)} + \mathfrak{F}_{(0,2,1,0,1)} + \mathfrak{F}_{(0,2,2,0,0)} + \mathfrak{F}_{(1,1,2,0,0)} + \mathfrak{F}_{(1,2,1,0,0)}. \tag{2.6}$$

This significantly compacts the 25-term monomial expansion of $\mathfrak{S}_{135264}$. Theorem 2.4 generalizes to Schubert polynomials Gessel's fundamental quasisymmetric expansion [8] of Schur polynomials.

The fundamental slide basis also has a triangularity with respect to the Schubert basis, allowing computationally efficient changes between these bases. Let $L(w)$ denote the Lehmer code of w, i.e., $L(w)_i$ is the number of $j > i$ such that $w_i > w_j$.

Proposition 2.5. *([2]) For w any permutation, there are coefficients $c_{w,b} \in \mathbb{Z}_{\geqslant 0}$ such that*

$$\mathfrak{S}_w = \mathfrak{F}_{L(w)} + \sum_{b > L(w)} c_{w,b} \mathfrak{F}_b. \tag{2.7}$$

We now consider stability properties of the fundamental slide expansion. For w a permutation of n, let $1^m \times w$ be the permutation $(1, \ldots, m, w(1)+m, \ldots w(n)+m)$ of $n+m$, and for a a weak composition of length n, let $0^m \times a$ be the weak composition of length $n+m$ obtained by prepending m zeros to a. Let $\mathrm{R}(w)$ be the set of reduced decompositions of w. One can define an explicit statistic η on permutations ([2]), indexing the precise point at which the fundamental slide expansion of $\mathfrak{S}_w$ stabilizes.

Theorem 2.6. *([2]) For w a permutation, if $\eta(w) \leqslant 0$, then $\#\mathrm{QPD}(w) = \#\mathrm{R}(w)$, and otherwise*

$$0 < \#\mathrm{QPD}(w) < \cdots < \#\mathrm{QPD}(1^{\eta(w)} \times w) = \cdots = \#\mathrm{R}(w). \tag{2.8}$$

In particular, the fundamental slide expansion of a Schubert polynomial is stable precisely when its terms are in bijection with $\mathrm{R}(w)$.

Corollary 2.7. *([2]) For any permutation w, let $\eta = \eta(w)$. Then, for any $m \geqslant \eta$, we have*

$$\mathfrak{S}_{1^m \times w} = \sum_a [\mathfrak{F}_a \mid \mathfrak{S}_{1^\eta \times w}] \mathfrak{F}_{0^{m-\eta} \times a}, \tag{2.9}$$

where $[\mathfrak{F}_a \mid \mathfrak{S}_{1^\eta \times w}]$ means the coefficient of $\mathfrak{F}_a$ in the fundamental slide expansion of $\mathfrak{S}_{1^\eta \times w}$.

For example, we have

$$\begin{aligned}
\mathfrak{S}_{24153} &= \mathfrak{F}_{(1,2,0,1)} + \mathfrak{F}_{(2,1,0,1)} + \mathfrak{F}_{(2,2,0,0)}, \\
\mathfrak{S}_{135264} &= \mathfrak{F}_{(0,1,2,0,1)} + \mathfrak{F}_{(0,2,1,0,1)} + \mathfrak{F}_{(0,2,2,0,0)} + \mathfrak{F}_{(1,1,2,0,0)} + \mathfrak{F}_{(1,2,1,0,0)}, \\
\mathfrak{S}_{1246375} &= \mathfrak{F}_{(0,0,1,2,0,1)} + \mathfrak{F}_{(0,0,2,1,0,1)} + \mathfrak{F}_{(0,0,2,2,0,0)} + \mathfrak{F}_{(0,1,1,2,0,0)} + \mathfrak{F}_{(0,1,2,1,0,0)}.
\end{aligned}$$

and the fundamental slide expansion of $\mathfrak{S}_{1^m \times 24153}$ has 5 terms for all $m \geqslant 1$.

3 Products

We give a combinatorial formula for the structure constants of the fundamental slide basis by generalizing the shuffle product of Eilenberg and Mac Lane [6] to weak compositions. Let $\varnothing$ denote the empty word.

Definition 3.1 ([6])**.** The *shuffle product* of words A and B, denoted by $A \sqcup\!\sqcup B$, is defined recursively by

$$A \sqcup\!\sqcup \varnothing = \varnothing \sqcup\!\sqcup A = \{A\} \text{ and } A \sqcup\!\sqcup B = \{A_1(A_2 \cdots A_{\ell(A)} \sqcup\!\sqcup B)\} \cup \{B_1(A \sqcup\!\sqcup B_2 \cdots B_{\ell(B)})\},$$

That is, $A \sqcup\!\sqcup B$ is the set of all ways of riffle shuffling the terms of A, in order, with the terms of B, in order. For example, we have

$$55111 \sqcup\!\sqcup 82 = \left\{ \begin{array}{llllll} 5511182 & 5511812 & 5518112 & 5581112 & 5851112 & 8551112 \\ 5511821 & 5518121 & 5581121 & 5851121 & 8551121 & 5518211 \\ 5581211 & 5851211 & 8551211 & 5582111 & 5852111 & 8552111 \\ 5825111 & 8525111 & 8255111 & & & \end{array} \right\}.$$

The *descent composition of* C, denoted by $\mathrm{Des}(C)$, is the lengths of successive increasing runs of the letters read from left to right. For the example above, the last three terms on the right hand side have descent compositions $(2,2,3), (1,1,2,3), (1,3,3)$, respectively.

Definition 3.2. ([2]) Let a, b be weak compositions of length n. Let A and B be the words defined by $A = (2n-1)^{a_1} \cdots (3)^{a_{n-1}} (1)^{a_n}$ and $B = (2n)^{b_1} \cdots (4)^{b_{n-1}} (2)^{b_n}$, and let $\mathrm{Des}_A(C)_i$ (respectively $\mathrm{Des}_B(C)_i$) be the number of letters from A (respectively B) in the ith increasing run of C. Define the *shuffle set of a and b*, denoted by $\mathrm{SS}(a,b)$, by

$$\mathrm{SS}(a,b) = \{C \in A \sqcup\!\sqcup B \mid \mathrm{Des}_A(C) \geqslant a \text{ and } \mathrm{Des}_B(B) \geqslant b\}. \tag{3.1}$$

For example, $\mathrm{SS}((0,2,0,3),(1,0,0,1))$ is given by

$$\mathrm{SS}((0,2,0,3),(1,0,0,1)) = \left\{ \begin{array}{ccccc} 5581112 & 5851112 & 8551112 & 5581121 & 5851121 \\ 8551121 & 5581211 & 5851211 & 8551211 & 5582111 \\ 5852111 & 8552111 & 5825111 & 8255111 & \end{array} \right\}.$$

Definition 3.3. ([2]) For weak compositions a, b of length n, define the *slide product of a and b*, denoted by $a \sqcup b$, to be the formal sum

$$a \sqcup b = \sum_{C \in \mathrm{SS}(a,b)} \mathrm{Des}(\mathrm{bump}_{(a,b)}(C)) \tag{3.2}$$

where $\mathrm{bump}_{(a,b)}(C)$ is the unique element of $0^{n-\ell(\mathrm{Des}(C))} \sqcup C$ with $\mathrm{Des}_A(\mathrm{bump}_{(a,b)}(C)) \geqslant a$ and $\mathrm{Des}_B(\mathrm{bump}_{(a,b)}(C)) \geqslant b$ and if $D \in 0^{n-\ell} \sqcup C$ satisfies $\mathrm{Des}_A(D) \geqslant a$ and $\mathrm{Des}_B(D) \geqslant b$, then $\mathrm{Des}(D) \geqslant \mathrm{Des}(\mathrm{bump}_{(a,b)}(C))$.

Continuing with our example, we have

$$\begin{aligned} (0,2,0,3) \sqcup (1,0,0,1) \quad = \quad & (3,0,0,4) + (2,1,0,4) + (1,2,0,4) + (3,0,3,1) + (2,1,3,1) \\ & (1,2,3,1) + (3,0,2,2) + (2,1,2,2) + (1,2,2,2) + (3,0,1,3) \\ & (2,1,1,3) + (1,2,1,3) + (2,2,0,3) + (1,3,0,3). \end{aligned}$$

Theorem 3.4. *([2]) For weak compositions a and b of length n, we have*

$$\mathfrak{F}_a \mathfrak{F}_b = \sum_c [c \mid a \sqcup b] \mathfrak{F}_c, \tag{3.3}$$

where $[c \mid a \sqcup b]$ means the coefficient of c in the slide product $a \sqcup b$.

From the running example, we have

$$\begin{aligned} \mathfrak{F}_{(0,2,0,3)} \mathfrak{F}_{(1,0,0,1)} \quad = \quad & \mathfrak{F}_{(3,0,0,4)} + \mathfrak{F}_{(2,1,0,4)} + \mathfrak{F}_{(1,2,0,4)} + \mathfrak{F}_{(3,0,3,1)} + \mathfrak{F}_{(2,1,3,1)} \\ & \mathfrak{F}_{(1,2,3,1)} + \mathfrak{F}_{(3,0,2,2)} + \mathfrak{F}_{(2,1,2,2)} + \mathfrak{F}_{(1,2,2,2)} + \mathfrak{F}_{(3,0,1,3)} \\ & \mathfrak{F}_{(2,1,1,3)} + \mathfrak{F}_{(1,2,1,3)} + \mathfrak{F}_{(2,2,0,3)} + \mathfrak{F}_{(1,3,0,3)}. \end{aligned}$$

Since the Schubert polynomial $\mathfrak{S}_w$ represents the Schubert class of w in the cohomology of the flag manifold, the structure constants $c_{u,v}^w$ of the Schubert basis enumerate flags in a generic triple intersection of Schubert varieties. Thus these so-called *Littlewood–Richardson coefficients* are nonnegative integers. A fundamental problem in Schubert calculus is to find a *positive* combinatorial construction for these numbers. One impediment to solving this problem is that computations quickly become intractable when multiplying out monomials. The following Littlewood–Richardson rule for the fundamental slide expansion of the product of Schubert polynomials gives us a more compact formula that should make computer experimentation possible.

Theorem 3.5. *([2]) For u, v permutations and a a weak composition, define $c^a_{u,v}$ by*

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_a c^a_{u,v} \mathfrak{F}_a. \tag{3.4}$$

Then we have

$$c^a_{u,v} = \sum_{(P,Q)\in \mathrm{QPD}(u)\times\mathrm{QPD}(v)} [a \mid \mathrm{wt}(P) \sqcup \mathrm{wt}(Q)]. \tag{3.5}$$

For example, we can compute the product $\mathfrak{S}_{24153}\mathfrak{S}_{2431}$ by

$$\begin{aligned}
\mathfrak{S}_{24153}\mathfrak{S}_{2431} &= \left(\mathfrak{F}_{(1,2,0,1)} + \mathfrak{F}_{(2,1,0,1)} + \mathfrak{F}_{(2,2,0,0)}\right)\left(\mathfrak{F}_{(1,2,1,0)} + \mathfrak{F}_{(2,1,1,0)}\right) \\
&= \mathfrak{F}_{(2,4,1,1)} + 2\mathfrak{F}_{(3,3,1,1)} + \mathfrak{F}_{(4,2,1,1)} + \mathfrak{F}_{(2,4,2,0)} \\
&\quad +2\mathfrak{F}_{(3,3,2,0)} + \mathfrak{F}_{(4,2,2,0)} + \mathfrak{F}_{(3,4,1,0)} + \mathfrak{F}_{(4,3,1,0)} \\
&= \left(\mathfrak{F}_{(2,4,1,1)} + \mathfrak{F}_{(3,3,1,1)} + \mathfrak{F}_{(4,2,1,1)}\right) + \left(\mathfrak{F}_{(3,3,1,1)}\right) + \left(\mathfrak{F}_{(3,3,2,0)}\right) \\
&\quad + \left(\mathfrak{F}_{(2,4,2,0)} + \mathfrak{F}_{(3,3,2,0)} + \mathfrak{F}_{(4,2,2,0)}\right) + \left(\mathfrak{F}_{(3,4,1,0)} + \mathfrak{F}_{(4,3,1,0)}\right) \\
&= \mathfrak{S}_{362415} + \mathfrak{S}_{45231} + \mathfrak{S}_{45312} + \mathfrak{S}_{364125} + \mathfrak{S}_{462135}.
\end{aligned}$$

Here, in the last step we made use of the triangularity between the Schubert basis and the fundamental slide basis given in Proposition 2.5.

Moreover, similarly to Theorem 2.6, one can define an explicit statistic ζ on pairs of permutations ([2]) that indexes the precise point at which the fundamental slide expansion of a product of Schubert polynomials stabilizes.

Theorem 3.6. *([2]) For permutations u, v, let $\zeta = \zeta(u, v)$. Then for all $m \geqslant \zeta$, we have*

$$\mathfrak{S}_{1^m\times u}\mathfrak{S}_{1^m\times v} = \sum_a [\mathfrak{F}_a \mid \mathfrak{S}_{1^\zeta\times u}\mathfrak{S}_{1^\zeta\times v}]\mathfrak{F}_{0^{m-\zeta}\times a}. \tag{3.6}$$

4 Kohnert tableaux and key polynomials

The key polynomials, originally defined by Demazure [5] and studied combinatorially by Lascoux and Schützenberger [12], form another important $\mathbb{Z}$-basis for polynomials. There are many combinatorial definitions of key polynomials: see, e.g., [14]. We will use a model due to Kohnert [10].

A *diagram* is an array of finitely many cells in the first quadrant of $\mathbb{Z}\times\mathbb{Z}$. Define the *key diagram of a* to be the diagram with a_i cells in row i, left justified. A *Kohnert move* on a diagram moves the rightmost cell of a given row to the first available position below, jumping over other cells in its way as needed. Let $\mathrm{KM}(a)$ be the set of all diagrams that can be obtained by applying a series of Kohnert moves to the key diagram of a. For example, see Figure 4.

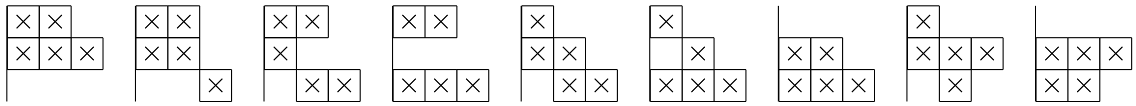

Figure 4: The nine Kohnert diagrams for $(0,3,2)$.

Definition 4.1 ([10]). The key polynomial indexed by a, denoted by κ_a, is given by

$$\kappa_a = \sum_{D \in \mathrm{KM}(a)} x^{\mathrm{wt}(D)}, \tag{4.1}$$

where $\mathrm{wt}(D)$ is the weak composition whose ith part gives the number of cells in row i.

For example, from Figure 4 we can compute

$$\kappa_{032} = x^{032} + x^{122} + x^{212} + x^{302} + x^{221} + x^{311} + x^{320} + x^{131} + x^{230}.$$

Definition 4.2. ([1]) Given a weak composition a of length n, a *Kohnert tableau of shape a* is a diagram filled with entries $1^{a_1}, 2^{a_2}, \ldots, n^{a_n}$, one per cell, satisfying:

(i) there is exactly one i in each column 1 through a_i;
(ii) each entry in row i is at least i;
(iii) the i's weakly descend from left to right;
(iv) if $i < j$ appear in a column with i above j, then there is an i right of and strictly above j.

Denote the set of Kohnert tableaux of shape a by $\mathrm{KT}(a)$.

For example, Figure 5 shows the elements of $\mathrm{KT}(0,3,2)$. Compare this with Figure 4.

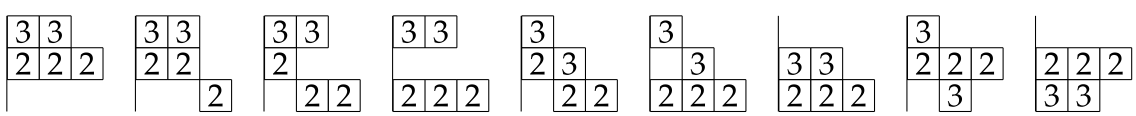

Figure 5: The nine Kohnert tableaux of shape $(0,3,2)$.

Theorem 4.3. *([1]) There is a weight-preserving bijection between* $\mathrm{KM}(a)$ *and* $\mathrm{KT}(a)$. *Thus,*

$$\kappa_a = \sum_{T \in \mathrm{KT}(a)} x^{\mathrm{wt}(T)}, \tag{4.2}$$

where $\mathrm{wt}(T)$ *is the weak composition whose ith part is the number of cells in row i of T.*

Definition 4.4. ([1]) A Kohnert tableau is *quasi-Yamanouchi* if the leftmost cell in each nonempty row i either has entry equal to i, or is weakly left of some cell in row $i+1$. Denote the set of quasi-Yamanouchi Kohnert tableaux of shape a by $\mathrm{QKT}(a)$.

For example, Figure 6 gives the quasi-Yamanouchi Kohnert tableaux of shape $(0,3,2)$.

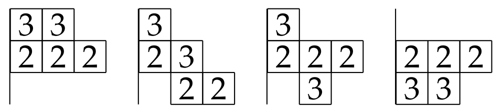

Figure 6: The four quasi-Yamanouchi Kohnert tableaux of shape $(0,3,2)$.

Similarly to Theorem 2.4, quasi-Yamanouchi Kohnert tableaux index the fundamental slide expansion of a key polynomial.

Theorem 4.5. *([1]) For a weak composition a, we have*

$$\kappa_a = \sum_{T \in \mathrm{QKT}(a)} \mathfrak{F}_{\mathrm{wt}(T)}, \tag{4.3}$$

where $\mathrm{wt}(T)$ is the weak composition whose ith part is the number of cells in row i of T.

Continuing the running example, from Figure 6 we can compute

$$\kappa_{032} = \mathfrak{F}_{032} + \mathfrak{F}_{221} + \mathfrak{F}_{131} + \mathfrak{F}_{230}.$$

Similarly to Theorem 2.6, one can define an explicit statistic η' on weak compositions ([1]), indexing the precise point at which the fundamental slide expansion of κ_a stabilizes. Let $\mathrm{sort}(a)$ be the entries of a arranged in decreasing order, and $\mathrm{SYT}(\mathrm{sort}(a))$ the set of standard Young tableaux of shape $\mathrm{sort}(a)$.

Theorem 4.6. *([1]) For a composition a of length n, and $m \geqslant \eta'(a)$, we have*

$$0 <^{\#} \mathrm{QKT}(a) < \cdots <^{\#} \mathrm{QKT}(0^{\eta'(a)} \times a) = \cdots =^{\#} \mathrm{QKT}(0^m \times a) =^{\#} \mathrm{SYT}(\mathrm{sort}(a)). \tag{4.4}$$

Corollary 4.7. *([1]) For any weak composition a, let $\eta' = \eta'(a)$. Then, for any $m \geqslant \eta'$, we have*

$$\kappa_{0^m \times a} = \sum_b [\mathfrak{F}_b \mid \kappa_{0^{\eta'} \times a}] \mathfrak{F}_{0^{m-\eta'} \times b}. \tag{4.5}$$

From Figures 6 and 7, we compute

$$\begin{aligned} \kappa_{32} &= \mathfrak{F}_{32} \\ \kappa_{032} &= \mathfrak{F}_{032} + \mathfrak{F}_{221} + \mathfrak{F}_{131} + \mathfrak{F}_{230} \\ \kappa_{0032} &= \mathfrak{F}_{0032} + \mathfrak{F}_{0221} + \mathfrak{F}_{0131} + \mathfrak{F}_{0230} + \mathfrak{F}_{1220}, \end{aligned}$$

and $|\mathrm{QKT}(0^m \times (3,2))| = 5$ for any $m \geqslant 2$.

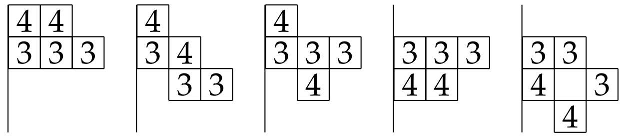

Figure 7: The five quasi-Yamanouchi Kohnert tableaux of shape $(0,0,3,2)$.

5 Quasi-key polynomials

We now impose additional conditions on Kohnert tableaux that will provide a combinatorial model for a new family of polynomials.

Definition 5.1. ([1]) Given a weak composition a of length n, a *quasi-Kohnert tableau of shape a* is a Kohnert tableau of shape a satisfying the following additional conditions:

(i) the leftmost column is strictly increasing from bottom to top, and
(ii) if $i < j$ are in consecutive columns with i left of and weakly above j, then $a_i \geqslant a_j$.

Denote the set of quasi-Kohnert tableaux of shape a by $\mathrm{qKT}(a)$.

For example, only the first eight Kohnert tableaux in Figure 5 satisfy the quasi-Kohnert conditions for the shape $(0,3,2)$.

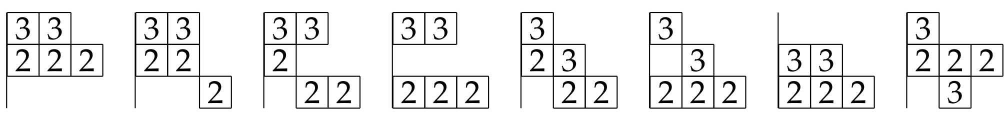

Figure 8: Quasi-Kohnert tableaux of shape $(0,3,2)$.

We now define quasi-key polynomials as the weighted sum of quasi-Kohnert tableaux.

Definition 5.2. ([1]) The *quasi-key polynomial indexed by a*, denoted by $\mathfrak{Q}_a$, is given by

$$\mathfrak{Q}_a = \sum_{T \in \mathrm{qKT}(a)} x^{\mathrm{wt}(T)}. \tag{5.1}$$

For example, from Figure 8, we compute

$$\mathfrak{Q}_{032} = x^{032} + x^{122} + x^{212} + x^{302} + x^{221} + x^{311} + x^{320} + x^{131}.$$

The quasi-key polynomials also expand positively in the fundamental slide basis, with the expansion indexed by the set $\mathrm{QqKT}(a)$ of quasi-Kohnert tableaux of shape a that are also quasi-Yamanouchi.

Theorem 5.3. *([1]) For a weak composition a of length n, we have*

$$\mathfrak{Q}_a = \sum_{T \in \mathrm{QqKT}(a)} \mathfrak{F}_{\mathrm{wt}(T)}. \tag{5.2}$$

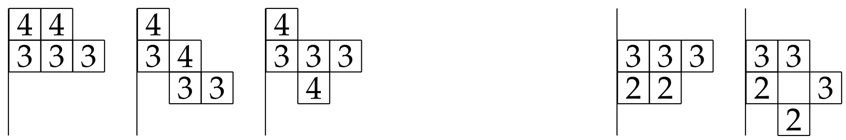

Figure 9: Quasi-Yamanouchi quasi-Kohnert tableaux of shape $(0,0,3,2)$ (left three) and of shape $(0,2,3,0)$ (right two).

For example, we compute the fundamental slide expansion for the quasi-key polynomials for $(0,0,3,2)$ and $(0,2,3,0)$ using the quasi-Yamanouchi Kohnert tableaux that satisfy the quasi-Kohnert conditions as depicted in Figure 9. In this case, we have

$$\mathfrak{Q}_{0032} = \mathfrak{F}_{0032} + \mathfrak{F}_{0221} + \mathfrak{F}_{0131} \qquad \text{and} \qquad \mathfrak{Q}_{0230} = \mathfrak{F}_{0230} + \mathfrak{F}_{1220}.$$

Recalling Figure 7, observe that $\mathfrak{Q}_{0032} + \mathfrak{Q}_{0230} = \kappa_{0032}$.

Key polynomials expand positively in quasi-key polynomials. A *left swap* on a weak composition a exchanges two parts $a_i < a_j$ of a where $i < j$. Let $\mathrm{lswap}(a)$ be the set of weak compositions obtainable via some (possibly empty) sequence of left swaps on a, and $\mathrm{Qlswap}(a)$ the dominance-minimal elements of $\mathrm{lswap}(a)$.

Theorem 5.4. *([1]) For any weak composition a,*

$$\kappa_a = \sum_{b \in \mathrm{Qlswap}(a)} \mathfrak{Q}_b. \tag{5.3}$$

By Theorem 5.4, we have the following corollary to Theorem 4.6:

Theorem 5.5. *([1]) For any weak composition a and for any $m \geqslant \eta'(a)$, we have*

$$\mathfrak{Q}_{0^m \times a} = \sum_b [\mathfrak{F}_b \mid \mathfrak{Q}_{0^{\eta'} \times a}] \mathfrak{F}_{0^{m-\eta'} \times b}. \tag{5.4}$$

For example,

$$\begin{aligned} \mathfrak{Q}_{32} &= \mathfrak{F}_{32} \\ \mathfrak{Q}_{032} &= \mathfrak{F}_{032} + \mathfrak{F}_{221} + \mathfrak{F}_{131} \end{aligned}$$

and $\mathfrak{Q}_{0^m \times 32}$ has three terms for $m \geqslant 1$.

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